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Is the Cromwell current driven by equatorial Rossby waves?

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A second-order theory for equatorial, baroclinic Rossby waves in a bounded ocean yields, among other components, terms of zero frequency. We inquire whether the associated *rectified* flow (or 'streaming', or 'mass transport') can be related to the submerged equatorial jet observed by Cromwell, Knauss and others. For quite reasonable models of baroclinic Rossby waves, the resultant streaming can be sharply concentrated at the equator, and varies in depth as $N(z)$, thus giving maximum flow at the thermocline (where the Väisälä frequency $N(z)$ is largest). In this way the rectification hypothesis might account for the observed submerged equatorial steering of the Cromwell current. The observed direction of streaming calls for a predominance of equatorially symmetric Rossby waves. The observed magnitude implies that the r.m.s. fluctuations be of the same order as the mean current; that is to say, we have applied weak-interaction theory to a strong-interaction problem. There are other difficulties.

1. Introduction

The equatorial undercurrent, discovered by Cromwell in 1951, flows eastward precisely along the equator. In the Pacific, where the Cromwell current is best developed, maximum velocities of the order of 1 m/sec are found at 100 m depths; the total transport of 4×10^{13} g/sec is comparable to that of the Gulf Stream. Hydrodynamicists are fascinated with the peculiar nature of the submerged equatorial steering of the Cromwell current;‡ we shall add yet another to the existing multitude of theories.

The formalism can be sketched as follows: equatorial Rossby waves in an unbounded ocean can be characterized by a two-dimensional discrete set (r, n) of up–down and north–south mode numbers, and by the dispersion relations $s = s^{\pm}_{rn}(\sigma)$ between east–west wave-numbers (s^{+} corresponds to posi-

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‡ Volume 6(4) of *Deep-Sea Research* (1960) is devoted to a review of observations and theories of the Cromwell current. See also Robinson (1966). Knauss (1966) reviews recent evidence.

tive (west-to-east) *group* velocity) and frequency. In the linear theory, the associated oscillatory currents $\mathbf{q}^{(1)}(u,v)$ can be written as a superposition of modes,

$$\mathbf{q}^{(1)}(x,y,z,t) = \int_0^\infty d\sigma \sum_{\pm rn} \mathbf{q}^{\pm}_{rn\sigma}\,\mathbf{K}^{(1)\pm}_{rn\sigma} \exp i[s^{\pm}x - \sigma t],$$

all equatorially trapped, where $s^{\pm}$ stands for $s^{\pm}_{rn}(\sigma)$, and $\sum_{\pm}$ implies summation over both signs. The vector $\mathbf{K}^{(1)}(y,z)$ is a linear transfer function. For an unbounded ocean, the $q^{\pm}_{rn\sigma}$ are independent mode amplitudes; for an ocean bounded by east and west coasts and with r, σ specified, the $q^{\pm}_{n}$ are all determined in terms of two arbitrary amplitudes.

Second-order theory yields currents $\mathbf{q} = \mathbf{q}^{(1)} + \mathbf{q}^{(2)}$, with

$$\mathbf{q}^{(2)} = \iint_0^\infty d\sigma' d\sigma'' \sum_{(rn\sigma)'}^{(\pm)'} \mathbf{q}'^{\pm}_{rn\sigma} \sum_{(rn\sigma)''}^{(\pm)''} \mathbf{q}''^{\pm}_{rn\sigma} \mathbf{K}^{(2)(\pm)'(\pm)''}_{(rn\sigma)'(rn\sigma)''} \exp i[(s(\sigma') \pm s(\sigma''))\,x - (\sigma' \pm \sigma'')t],$$

where $s(\sigma')$ stands for $s^{(\pm)'}_{(rn)'}$, and similarly for $s(\sigma'')$, and where $\mathbf{K}(y,z)$ are quadratic coupling matrices, to be summed (for each $r'n'\sigma'r''n''\sigma''$) over all possible eight sign combinations. The part of $\mathbf{q}^{(2)}$ corresponding to a zero difference frequency, $\sigma' - \sigma'' = 0$, to be denoted by $\mathbf{Q}(x,y,z)$, is associated with 'rectified' currents. In the case of a single, self-interacting r-mode, it turns out that

$$\mathbf{Q} = \operatorname{curl} \psi \quad \text{with} \quad \psi(x,y,z) = \langle u \textstyle\int v\,dt\rangle + F(y,z),$$

so that, apart from an arbitrary function $F(y,z)$, the (rectified) streamfunction is given by the mean quadrature product of the orbital components.

2. Wave equations

The linearized wave equations in a vertically stratified, incompressible, equatorial ocean are

$$u_t - fv = -p_x, \quad v_t + fu = -p_y, \tag{1, 2}$$

$$w_t = -p_z + \phi, \quad \phi_t = -N^2(z)w, \tag{3, 4}$$

$$u_x + v_y + w_z = 0, \tag{5}$$

where density $= \rho_0(x,y) + \hat{\rho}(z) + \rho(x,y,z,t)$,

$$p = \frac{\text{pressure}}{\rho_0}, \quad \phi = -g\frac{\hat{\rho}+\rho}{\rho_0}, \quad N^2(z) = -g\frac{\hat{\rho}_z}{\rho_0}.$$

With the hydrostatic assumption, $\sigma \ll N$, the term w_t is negligible, and (3) and (4) are combined into

$$wN^2(z) + p_{zt} = 0. \tag{6}$$

3. Kelvin waves

For illustration we shall sketch the degenerate case of a wave motion for which $v = 0$; the example is somewhat misleading, as we shall see. The solutions

$$u = \pm q \cos(sx \mp \sigma t)\, e^{\mp\alpha^2 y^2} Z_z, \quad v = 0, \tag{7}$$

$$p = cq \cos(sx \mp \sigma t)\, e^{\mp\alpha^2 y^2} Z_z$$

satisfy (1) to (5), provided

$$c = \sigma/s, \quad \alpha^2 = \beta/2c, \quad f = \beta y.$$

For equatorially trapped waves we must choose the upper sign; hence the waves travel in a positive (eastward) direction, with velocity c.

The function $Z(z)$ (proportional to the vertical displacement) must satisfy the differential equation

$$Z_{zz} + c^{-2}N^2(z)\,Z = 0 \tag{8}$$

subject to the bottom and surface boundary conditions

$$Z = 0 \quad \text{at} \quad z = 0, \qquad Z_z = gc^{-2}Z \quad \text{at} \quad z = h.$$

For a gradual variation in $N(z)$ we can use the W.K.B.J. approximation to (8):

$$Z(z) \sim N^{-\frac{1}{2}}(z)\sin\kappa z, \quad Z_z(z) \sim c^{-1}N^{\frac{1}{2}}(z)\cos\kappa z, \tag{9}$$

$$\kappa(z) = \frac{1}{cz}\int_0^z N(z)\,dz,$$

and so approximately

$$\left.\begin{aligned} c_0 &= \sqrt{(gh)} \approx 200\,\text{m/sec},\\ c_r &= \frac{h\overline{N}}{r\pi} \approx \frac{2{\cdot}30}{r}\,\text{m/sec} \quad (r = 1, 2, \ldots), \end{aligned}\right\} \tag{10}$$

for external and internal modes, where $\overline{N} = h^{-1}\int_0^h N\,dz \approx 25$ cyc./day is a typical value for the Väisälä frequency averaged from bottom to top in a 4 km ocean.

We now write

$$u(x,y,z,t) = q_r \cos(sx - \sigma t)\,e^{-\alpha^2 y^2}[N(z)/\overline{N}]^{\frac{1}{2}}\cos\kappa z$$

for the orbital velocity in a Kelvin wave, where q_n is a representative amplitude for the rth mode.

The time-averaged Bernoulli pressure is simply

$$P(y,z) = -\tfrac{1}{2}\langle u^2\rangle = -\tfrac{1}{4}q_r^2\,e^{-\beta y^2/c_r}[N(z)/\overline{N}]\cos^2\kappa z.$$

For high up–down modes (large $\kappa(z)$) we replace $\cos^2\kappa z$ by its (local) vertical average:

$$P(y,z) = -\tfrac{1}{8}q_r^2\,e^{-\beta y^2/c_r}N(z)/\overline{N}. \tag{11}$$

The current in geostrophic balance with this pressure distribution is given by

$$U(y,z) = -\frac{P_y}{f} = -\frac{1}{4}\frac{q_r^2}{c_r}\,e^{-\beta y^2/c_r}\frac{N(z)}{\overline{N}}, \tag{12}$$

and the total transport is

$$\int_{-\infty}^{\infty}\int_0^h U\,dy\,dz = -\tfrac{1}{4}q_r^2 h(\beta c_r)^{-\frac{1}{2}}.$$

This is not, by any means, a *derivation* of wave-induced streaming. But it turns out that the systematic development (which will follow in §5) reproduces most of the features of (12). Meanwhile, this preliminary discussion establishes the appropriate scales for the general case of planetary waves (§4), and provides a surprisingly good framework for summarizing the observational evidence. In particular:

(i) The vertical profile of $U(z)$ is determined by $N(z)$. This is a remarkably good prediction for the Cromwell current in the Western Pacific; further east where the thermocline is at greater depth, so is the current† (figure 1).

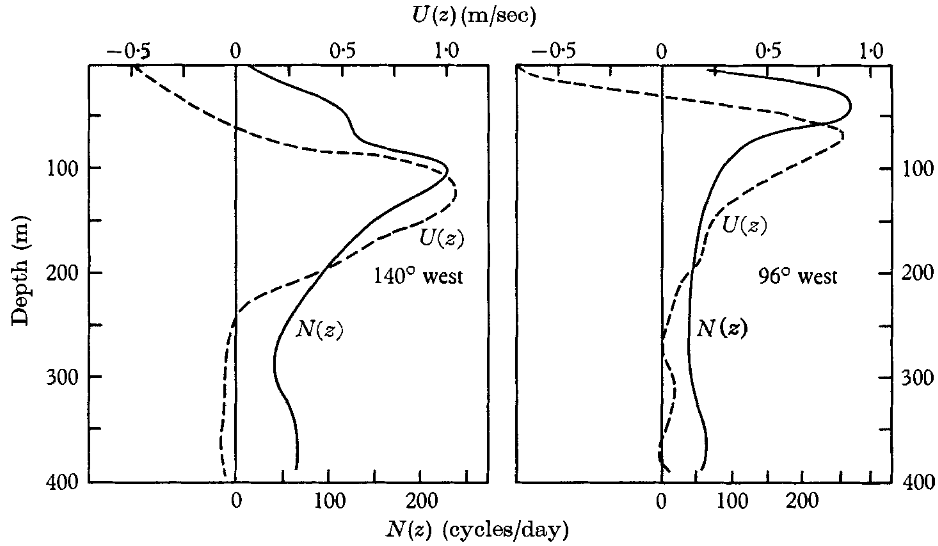

FIGURE 1. Vertical profiles at the equator of eastward velocity U and Väisälä frequency N (courtesy John Knauss).

(ii) The e-folding (or 'critical') latitude is given by $\beta y_c^2/c_r = 1$, hence

$$\phi_c = \frac{y_c}{a} = \pm\left(\frac{c_r}{2a\Omega}\right)^{\frac{1}{2}} \approx \pm(300r)^{-\frac{1}{2}},$$

or roughly $\pm 3°$ latitute for $r = 1$, and $\pm 1°$ for $r = 9$. Observed profiles‡ $U(y, z)$ indicate something like $\pm 2°$:

	2° S.	1° S.	0	1° N.	2° N.	
Depth 120 m	7	42	115	57	3	cm/sec
Depth 140 m	50	36	96	62	32	cm/sec

(iii) Mean flow and mean-square flow are related according to $U = \langle u^2 \rangle / c_r$; for large r (small c_r) we can have U, u, c_r all of the same order§, so that the current could be maintained if departures from the mean were comparable with the mean current itself. (For the surface (barotropic) mode, c_0 is very large and its contribu-

† A correction for the westward surface flow associated with the south-equatorial current (a shallow flow) would somewhat improve the resemblance between observed and computed flow.

‡ Expedition *Swan Song*, 140° W., by courtesy John Knauss.

§ This is clearly beyond the present perturbation treatment.

tion is absolutely negligible.) Knauss (1966) measured the Cromwell transport at 140° W. in October 1961 (expedition *Swan Song*) and found 55 % of the transport measured in April 1958 at the same place.

(iv) The reader will have noticed that the computed current flows in the wrong direction. The reason is simple: the increased mean-square velocity is associated with an equatorial pressure trough, and this implies geostrophic motion from east to west.

The question is, can something be done about the current direction without destroying the more pleasant features of the hypothesis? The answer is: Yes!

4. Planetary–gravity (pg) waves

Equatorial scaling

At this stage it is convenient to scale in terms of the north–south e-folding distance $(c/\beta)^{\frac{1}{2}}$. An associated non-dimensional parameter is the reciprocal square of the critical latitude:

$$\phi_c^{-2} = \gamma_r = \frac{2a\Omega}{c_r} = 2\pi\frac{a/n}{\tilde{N}}r \approx 10^{\frac{5}{2}}r \quad (r = 1, 2, \ldots), \tag{13}$$

where $\tilde{N} = \overline{N}_d/\Omega$ is the *mean* Väisälä frequency in cycles/day (c/d), and where $\frac{1}{2}r$ is the number of vertical cycles/ocean depth. This large parameter γ conveniently represents the relatively large vertical wave-numbers associated with baroclinic planetary waves. We now write

$$\xi = \gamma^{\frac{1}{2}}(x/a), \quad \eta = \gamma^{\frac{1}{2}}(y/a),$$
$$\zeta = \pi r(z/h) = \tfrac{1}{2}\gamma\tilde{N}(z/a), \quad \tau = 2\Omega\gamma^{-\frac{1}{2}}t,$$

for the eastward, northward and upward dimensionless co-ordinates, and for dimensionless time. We scale velocities and pressure according to

$$\begin{bmatrix} u_d(x,y,z,t) \\ v_d(x,y,z,t) \\ p_d(x,y,z,t) \end{bmatrix} = qZ_\zeta \begin{bmatrix} u(\xi,\eta,\tau) \\ v(\xi,\eta,\tau) \\ cp(\xi,\eta,\tau) \end{bmatrix}, \tag{14}$$

where
$$Z_\zeta = dZ/d\zeta = [\tilde{N}(z)/\tilde{N}]^{\frac{1}{2}}\cos\kappa z,$$
$$\kappa(z)z = c^{-1}\int_0^z N\,dz,$$

is the short-wave solution to the differential equation

$$\tilde{N}^2 Z_{\zeta\zeta} + \tilde{N}^2(z)\,Z = 0, \tag{15}$$

as previously found ($\tilde{N}(z)$ as $\tilde{N}$ are the variable and mean Väisälä frequencies in c/d). The surface-boundary condition quantizes γ_r (equation (13)). The system of equatorial equations now becomes

$$\left.\begin{aligned} u_\tau - \eta v + p_\xi &= 0, \\ v_\tau + \eta u + p_\eta &= 0, \\ u_\xi + v_\eta + p_\tau &= 0. \end{aligned}\right\} \tag{16}$$

The scaling involves the vertical mode number r. This is an important limitation to the present approach.

Planetary–gravity waves

Progressive pg-wave solutions can be written

$$\begin{bmatrix} u(\xi,\eta,\tau) \\ v(\xi,\eta,\tau) \\ p(\xi,\eta,\tau) \end{bmatrix} = e^{i(s\xi-\sigma\tau)} \begin{bmatrix} Y^u \\ iY \\ Y^p \end{bmatrix} \tag{17}$$

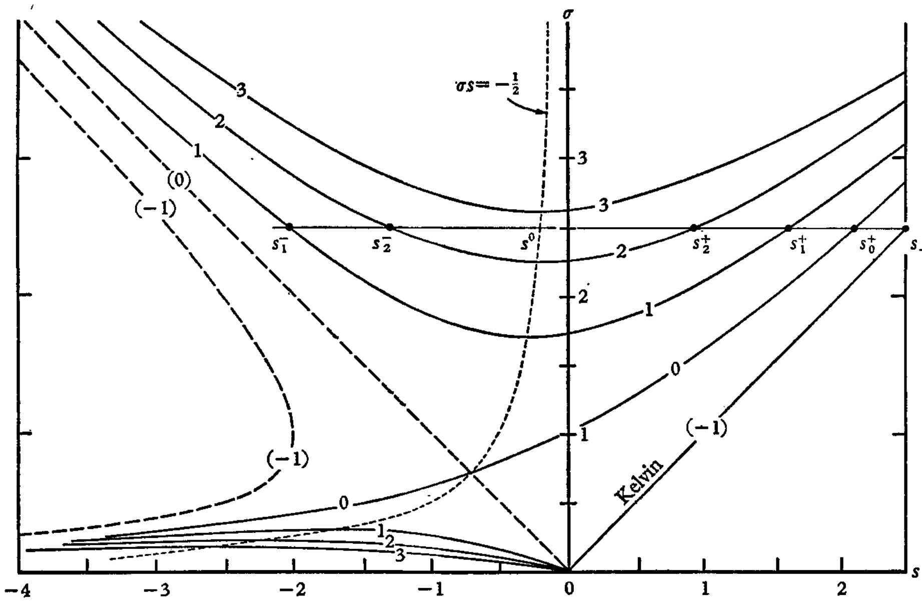

FIGURE 2. The planetary–gravity wave dispersion $s_n(\sigma)$ for indicated values of n. The dashed curves for $n = 0, -1$, are not admissible dispersions in the context of equatorial trapping. Positive longitudinal wave-numbers s correspond to west-to-east wave propagation with phase velocity σ/s; positive slope $\partial\sigma/\partial s$ to west-to-east group velocity. Planetary (or Rossby) wave solutions lie beneath $n = 0$ for negative s; gravity wave solutions lie above $n = 0$. All admissible real roots $s_n^{\pm}(\sigma) \equiv s^0(\sigma) \pm s_n'(\sigma)$ for the case $\sigma = 2{\cdot}5$ are labelled.

provided the non-dimensional (spacial and temporal) frequencies,

$$s = \gamma^{-\frac{1}{4}} a s_d, \quad \sigma = \tfrac{1}{2}\gamma^{\frac{1}{4}}\sigma_d/\Omega, \tag{18}$$

satisfy the pg-dispersion relation (see figure 2)

$$2n+1-\sigma^2+s^2+s/\sigma = 0, \tag{19}$$

and provided $Y_n(\eta)$ obeys the Weber equation

$$Y_{\eta\eta}+[(2n+1)-\eta^2]\,Y = 0; \quad n \text{ any real number.} \tag{20}$$

The associated functions $Y^u(\eta)$ and $Y^p(\eta)$ are given by

$$Y^u = -\frac{\sigma\eta Y - sY_\eta}{\sigma^2-s^2}, \quad Y^p = \frac{\sigma Y_\eta - s\eta Y}{\sigma^2-s^2}. \tag{21}$$

The case $s = \pm\sigma$ will require special consideration.

We are here concerned with that subset of the Weber function $Y_n(\eta)$ which vanishes for $\eta = \pm\infty$:

$$Y_n(\eta) = (2^n n!)^{-\frac{1}{2}}\pi^{-\frac{1}{4}}\, e^{-\frac{1}{2}\eta^2} H_n(\eta) \quad (n = 0, 1, 2, \ldots), \tag{22}$$

where $H_n(\eta)$ is the nth Hermite polynomial. Negative† n integers are not equatorially trapped. For $n = 0$, equation (19) has the roots $s_0 = -\sigma$ and $s_0 = (\sigma^2 - 1)/\sigma$. The u-field associated with $s_0 = -\sigma$ is unbounded at high latitudes, and therefore inadmissible (Matsuno 1966). The Kelvin wave solution has the unique property $v = 0$ and does not fall into class (17), even though its dispersion relation $\sigma_d/s_d = c$ (or $\sigma = s$ in non-dimensional units) is one of the roots of (19) for the case $n = -1$, as pointed out by Matsuno. (The other root, $s_{-1} = -(\sigma + \sigma^{-1})$ is not admissible.) We refer to Matsuno (1966) and Blandford (1966) for further discussion.‡ The Blandford–Matsuno notation (18) has the advantage of bringing the dispersion (19) into a canonical form, and leading to the compact presentation of figure 2, but at the expense of camouflaging the dependence on the vertical mode structure parameter γ_r. We may visualize figure 2 as a plot of $\tilde{s}(2\tilde{\sigma})$ for $\gamma = 1$, where

$$\tilde{s} = a s_d = \gamma^{\frac{1}{2}} s, \quad \tilde{\sigma} = \sigma_d/\Omega = 2\gamma^{-\frac{1}{2}}\sigma, \tag{23}$$

are dimensionless frequencies (in cycles/earth circumference and cycles/day, respectively) that do not involve γ_r. In $(\tilde{\sigma}, \tilde{s})$-space all curves move downward (toward smaller $\tilde{\sigma}$) and outward (toward larger $|\tilde{s}|$) with increasing γ.

Equatorial confinement

For orientation, we inquire what *single* combination of r and n would have the appropriate latitude scale. $Y_n(\eta)$ has n zeroes between the turning 'latitudes' $\eta_r = \pm(2n+1)^{\frac{1}{2}}$ and sharp cut-offs beyond. The average spacing between successive zeroes is $2(2/n)^{\frac{1}{2}}$ in y-scale, or $2(2/n)^{\frac{1}{2}}\gamma^{-\frac{1}{2}}$ in latitude. We identify half this quantity with the estimated width ϕ_c of the Cromwell current:

$$\phi_c^2 = 2/(\gamma n), \quad \gamma \approx 300r, \quad \phi_c = 0{\cdot}02 \text{ radians;}$$

hence for	$n =$	1	3	9
we have	$\gamma =$	5000	1667	555
	$r =$	18	6	2
	$\phi_r =$	0·02	0·06	0·18 radians

to which we add

$\sigma^0 =$	0·29, 1·71	0·19, 2·63	0·11, 4·36
$\tilde{\sigma}^0 =$	0·0082, 0·048	0·0093, 0·13	0·0098, 0·37 c/d

for the maximum planetary and minimum gravity wave frequencies associated with the choice n, r (see figure 2). These are found by setting $\partial\sigma/\partial s = 0$ in (19); hence

$$s^0\sigma = -\tfrac{1}{2}, \quad (\sigma^0)^2 + (2\sigma^0)^{-2} = 2n + 1, \tag{24}$$

† Suitable definitions are given in Erdelyi *et al.* (1953, 8.3 (19) and 10.13 (7)).

‡ A presentation similar to figure 2 was given by Groves (1965).

as shown in figure 2 by the intersection of the n-lines with the dashed curve $s\sigma = -\frac{1}{2}$. The corresponding frequencies $\tilde{\sigma}^0 = 2\gamma^{-\frac{1}{2}}\sigma^0$ in c/d are given on the last line. At large n, $\tilde{\sigma}^0_{\text{Rossby}}$ approaches $\frac{1}{2}\phi_c$.

For low n, one requires quite high r-values in order for the current to be adequately concentrated. The work of Cox & Sandstrom (1962) and of Hendershott (unpublished Harvard University thesis) concerning the scattering of surface into internal tides by bottom irregularities does in fact favour quite high r-values, typically, $r = 10$. On the other hand, at subtidal frequencies (and these turn out to be the essential frequencies in the present context), Phillips (1966) finds little evidence of high vertical modes. None of the observations quoted here were equatorial.

In fact, the Cromwell current is not to be interpreted as the result of a single n, r mode, but rather from superposition of many modes. It turns out that $\phi_c \sim (\gamma n)^{-\frac{1}{2}}$ is still a meaningful representation provided n is interpreted as the *highest* Hermite order of any consequence.

The two-class approximation

The pg-solutions fall into two classes,

class 1: gravity waves,

class 2: planetary (or Rossby) waves,

as shown in figure 2. For these two classes the terms in equation (19), s/σ and $-\sigma^2$ respectively, play a secondary role, and can be neglected in the asymptotic cases of very high or very low Blandford–Matsuno frequencies. Referring now to (21) we find the u solution to depend on two latitude functions, ηY and Y_η, weighted in the ratio $\sigma : s$, and it would seen natural to ignore one or the other in the case of small or large σ. The standing planetary wave solutions of Rattray & Charnell (1966) are arrived at in this manner.

But the roots to the quadratic (19)

$$\left.\begin{aligned} s_n^{\pm} &= s^0 \pm s_n', \\ s^0 = -1/2\sigma, \quad (s_n')^2 &= (s^0)^2 + \sigma^2 - (2n+1), \end{aligned}\right\} \tag{25}$$

reduce to

$$s^+ = -(2n+1)\sigma, \quad s^- = -1/\sigma \quad \text{for} \quad \sigma \ll 1,$$

$$s^{\pm} = \pm\sigma \quad \text{for} \quad \sigma \gg 1,$$

respectively, and there is no justification for ignoring one or the other of the η-functions, at least not as far as the highly baroclinic (strongly divergent) situation is concerned. Moreover, our emphasis on the equatorial problem does not lead naturally to the two-class approximation for large or small σ; rather it is associated with small n-integers and hence σ of order 1.

The standing-wave problem

We shall need solutions in the presence of boundaries at $\xi = 0, \xi_L$, and here the questions raised in the preceding section play a vital role. We superimpose waves of positive and negative group velocity and require that

$$q^+u^+ + q^-u^- = 0$$

at $\xi = 0, \xi_L$. In the former case this implies that

$$q_n^+ Y_n^{u+} + q_n^- Y_n^{u-} = 0, \quad Y_n^{u\pm} = -\frac{\sigma\eta Y_n - s^\pm(Y_n)\eta}{\sigma^2 - (s_n^\pm)^2} \tag{26}$$

be satisfied at all latitudes. But, since Y^u is the sum of two linearly independent latitude functions, condition (26) needs to be satisfied for both separately; this can be done only if

$$\begin{vmatrix} 1/D_n^+ & 1/D_n^+ \\ s_n^+/D_n^+ & s_n^-/D_n^- \end{vmatrix} = 0, \quad D_n^\pm = \sigma^2 - (s_n^\pm)^2,$$

which leads to the generally unacceptable condition $s_n^+ = s_n^-$. We have shown that this difficulty cannot be avoided by asymptotically ignoring one or the other η-dependence.

The physical significance and the appropriate procedure have been developed by D. Moore and A. R. Robinson (D. Moore, Harvard University thesis, in preparation), and the remainder of this section summarizes the relevant results. In order to satisfy the boundary condition that $u = 0$ on $\xi = 0$ and $\xi = \xi_L$ it is necessary to use a superposition of all the available separable solutions corresponding to integer values of n. With the help of the recurrence relations

$$\begin{bmatrix} \eta Y_n \\ \dfrac{\partial}{\partial\eta} Y_n \end{bmatrix} = \begin{bmatrix} + \\ - \end{bmatrix} \left(\frac{n+1}{2}\right)^{\frac{1}{2}} Y_{n+1} + \left(\frac{n}{2}\right)^{\frac{1}{2}} Y_{n-1},$$

Moore writes the totality of separable solutions as

$$\begin{bmatrix} u_n^\pm(\xi,\eta,\tau) \\ v_n^\pm(\xi,\eta,\tau) \end{bmatrix} = \exp\{i(s_n^\pm \xi - \sigma\tau)\} \begin{bmatrix} \dfrac{-(\frac{1}{2}n)^{\frac{1}{2}} Y_{n-1}}{s_n^\pm + \sigma} + \dfrac{\{\frac{1}{2}(n+1)\}^{\frac{1}{2}} Y_{n+1}}{s_n^\pm - \sigma} \\ iY_n \end{bmatrix}$$

for $n = 1, 2, 3 \ldots$, plus the special cases

$$\begin{bmatrix} u_0(\xi,\eta,\tau) \\ v_0(\xi,\eta,\tau) \end{bmatrix} = \exp\{i(s_0\xi - \sigma\tau)\} \begin{bmatrix} -2^{-\frac{1}{2}}\sigma Y_1 \\ iY_0 \end{bmatrix}$$

for $n = 0$, where $s_0 = \sigma - \sigma^{-1}$, and

$$\begin{bmatrix} u_{-1}(\xi,\eta,\tau) \\ v_{-1}(\xi,\eta,\tau) \end{bmatrix} = i\exp\{i\sigma(\xi - \tau)\} \begin{bmatrix} Y_0 \\ 0 \end{bmatrix}$$

as a notation for the Kelvin waves. At a given frequency σ there are two solutions, one with u an odd function of η and v an even function of η, the other with u even and v odd. In the latter case we write

$$u = q_{-1}u_{-1} + \sum_{n=1,3,\ldots} (q_n^+ u_n^+ + q_n^- u_n^-), \quad v = \sum_{n=1,3,\ldots} (q_n^+ v_n^+ + q_n^- v_n^-)$$

and determine the $q_n^\pm$ by requiring that $u = 0$ on $\xi = 0, \xi_L$.

The coefficients are determined successively by invoking the completeness of the Hermite functions, and are given in terms of the arbitrary amplitude of the Kelvin wave q_{-1} by

$$q_1^{\pm} = \pm 2^{-\frac{1}{2}} q_{-1} \frac{s_1^{\pm}+\sigma}{\sin s_1' \xi_L} [\exp\{-i(s^0-\sigma)\xi_L\} - \exp\{\mp s_1' \xi_L\}],$$

$$q_n^{\pm} = \mp \frac{i}{2}\left(\frac{n-1}{n}\right)^{\frac{1}{2}} \frac{s_n^{\pm}+\sigma}{\sin s_n' \xi_L}\left[\frac{\exp\{is_{n-2}'\xi_L\} - \exp\{\mp is_n'\xi_L\}}{s_{n-2}'-\sigma} q_{n-2}^{+} + \frac{\exp\{-is_{n-2}'\xi_L\} - \exp\{\mp is_n'\xi_L\}}{s_{n-2}'-\sigma} q_{n-2}^{-}\right] \quad (27)$$

so long as $\sin s_n' \xi_L$ does not vanish for any n. If it does vanish for any n, let n' be the largest such value. In that case $q_n^{\pm} = 0$ for $n < n'$, and

$$q_{n'}^{-} = -\frac{s_{n'}^{-}+\sigma}{s_{n'}^{+}+\sigma} q_{n'}^{+}. \quad (28)$$

For $n > n'$, the $q_n^{\pm}$ are still given by (27), but are in terms of the arbitrary amplitude $q_n^{\pm}$, rather than q_{-1}. We shall call this the truncated case.

For the case where v is even in n and u odd, we write

$$u = q_0 u_0 + \sum_{n=2,4,\ldots} (q_n^+ u_n^+ + q_n^- u_n^-), \quad v = q_0 v_0 + \sum_{n=2,4,\ldots} (q_n^+ v_n^+ + q_n^- v_n^-), \quad (29)$$

and determine the coefficients $q_n^{\pm}$ in a manner completely analogous to the case discussed above.

For any given σ there is some maximum integer n (n'' say) beyond which there are no real solutions for s (equation (24)). The summation (27) extends indefinitely beyond n'' and so is associated with imaginary values $s'_{n>n''}$. These correspond to waves decaying exponentially from either boundary; they do not contribute to the interior solution. Moore associates this part of the solution with a poleward-travelling Kelvin wave along the eastern boundary, and an equator-ward wave along the western boundary. A northern and southern boundary requires that these waves be matched and so quantizes σ. This is not essentially involved in the equatorial problem.

We now have the apparatus for computing the non-linear stresses. To keep things manageable we shall choose n' and n'' such as to permit the simplest meaningful solutions.

5. Mean circulation

Equations of mean flow

Let $Q(U, V, W)$ and $q(u, v, w)$ designate the mean and oscillatory flow,† respectively, and P, p the associated pressures divided by ρ_0. The mean horizontal equations of motion are

$$f\mathbf{k} \times \mathbf{Q} = -\nabla_H P + \boldsymbol{\tau}_H, \quad (30)$$

where $$\boldsymbol{\tau}(x,y,z) = \langle -\mathbf{q}.\nabla\mathbf{q}\rangle$$

† All units are now dimensional.

is the mean† stress associated with the equatorial wave solutions. The terms $\mathbf{Q}.\nabla\mathbf{Q}$ are ignored, even though they can be demonstrated from observations to be comparable to the linear terms (Knauss 1966). This is in line with our perturbation treatment. We also have

$$WN^2(z) = [-\mathbf{q}.\nabla p_z] = \langle -\nabla(p_z\mathbf{q})\rangle \tag{31}$$

for the vertical motion, and

$$\nabla.\mathbf{Q} = 0. \tag{32}$$

Cross-differentiation of (30) leads to the equation

$$-fW_z + \beta V = \mathbf{k}.(\nabla\times\boldsymbol{\tau}). \tag{33}$$

For the case of vertically integrated transports the term W disappears, and we have Sverdrup's (1947) relation for the equatorial transports from specified surface wind stresses. Our problem is a generalization of Sverdrup's treatment in so far as the interior stresses are presumed to be known.

Geostrophy

Anticipating the result $W = 0$, equation (33) states that βV is of order τ/y_c, where $2y_c$ is the width of the Cromwell current. This immediately implies that, in the east–west component of the momentum equation (30), the fV term, is comparable to τ and geostrophy does not prevail. But with regard to the north–south component, since τ is of order fV, the fU term is larger than τ in the ratio $U:V$. U is observed to be much larger than V and we can expect near-geostrophic balance between fU and $-P_y$. These remarks are quite general, and apply to any realistic model based on equatorial vorticity balance. Knauss (1960), Metcalf, Voorhis & Stalcup (1962), Montgomery & Stroup (1962) and Reid (1964) found the Cromwell current and the observed density field related in accordance with the geostrophic equation. Surprisingly the opposite conclusion was reached by Knauss (1966) for expedition *Swan Song*.

Mean east–west pressure gradient

Knauss (1966) has measured a mean east–west pressure force, $-\overline{P}_x = 2{\cdot}6\times10^{-5}$ dynes/g, at the depth of the core. He takes the view, now widely held, that the Cromwell current flows simply in response to this horizontal pressure gradient. For our theory the mean east–west pressure gradient across the ocean vanishes (as will be shown), so this is a crucial point.

We are not convinced by the observational evidence for $\overline{P}_x$. The value quoted above refers to an average between 140° W. and 104° W. But, according to figure 16 of Knauss's paper, the mean gradients are essentially constant, or even reversed, if attention is focused to the region east of 130° W. and deeper than 75 m, including the sector at 118° W. where the observed flow was very pronounced.

† The time mean of any function $F(x, y, z, t)$ is defined by

$$\langle F\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} F(x, y, z, t)\,dt.$$

Writing $$u = u_c(\xi, \eta, y)\cos\sigma t + u_s(\xi, \eta, y)\sin\sigma t,$$

and similarly for v, w, p, it can be demonstrated (D. Moore, Harvard University thesis) that

$$u_c, v_s, p_c, w_s \text{ are odd functions of } \xi - \tfrac{1}{2}\xi_L,$$

$$u_s, v_c, p_s, w_c \text{ are even functions of } \xi - \tfrac{1}{2}\xi_L.$$

The x-momentum equation at the equator is $-P_x + \tau^{(x)} = 0$, and thus

$$\overline{P}_\xi = \frac{1}{\xi_L}\int_0^{\xi_L} \tau^{(x)}\,d\xi = -\frac{1}{\xi_L}\int_0^{\xi_L} \langle u u_\xi + v u_\eta + w u_\zeta\rangle\,d\xi.$$

But all products in $\langle\ \rangle$ are odd functionsof $\xi - \frac{1}{2}\xi_L$, and accordingly the integral must vanish. An equatorial isobar is at the same level at the two coasts. In mid-ocean it is lowered by $O(u^2/g)$; at core depth this is $O(10\,\text{cm})$, not necessarily incompatible with Knauss's figure 16.

Reynolds stresses and torques

We need to calculate $\langle\nabla.(p_z q)\rangle$ and $\langle\mathbf{k}.\nabla\times\boldsymbol{\tau}\rangle$. Direct substitution from the formal series solutions is tedious. The calculation may be greatly simplified by taking advantage of the following conditions:

(i) The fluctuating fields satisfy the equations of motion (16). Two formulations of (16) are particularly useful. The time average of the energy equation gives (dimensionless)

$$-\tfrac{1}{2}\langle(u^2+v^2+p^2)_t\rangle = \langle up\rangle_\xi + \langle vp\rangle_\eta = 0 \tag{34}$$

in view of (ii), and the vertical component of vorticity obeys the relation

$$\omega_t \equiv (v_x - u_y)_t = \eta p_t - v. \tag{35}$$

(ii) The time average of the time derivative of any bounded function is zero. This implies, for instance, that, if $f = gh$, where g and h are any bounded functions, then

$$\langle f_t\rangle = \langle g_t h + g h_t\rangle = 0, \quad \text{or} \quad \langle g_t h\rangle = -\langle g h_t\rangle. \tag{36}$$

(iii) The z-dependence is separable; and, for simplicity, we deal with a single vertical mode, cf. (14).

(iv) Ultimately we shall permit ourselves to form local vertical averages in the sense of (11).

Vertical flow

The expression (31) in terms of the non-dimensional variables of § 4 can be written

$$\begin{aligned} W &= [N(z)]^{-2}\langle\nabla\cdot(p_z q)\rangle_{\text{dimensional}} \\ &= \frac{q^2}{c}\frac{2}{\gamma^{\frac{1}{2}}\tilde{N}}\left\{\langle(up)_\xi + (vp)_\eta\rangle ZZ_\zeta + \left[\frac{\tilde{N}}{\tilde{N}(z)}\right]^2 \langle pp_t\rangle (ZZ_{\zeta\zeta})_\zeta\right\}. \end{aligned} \tag{37}$$

But $\langle(up)_\xi + (vp)_\eta\rangle = 0$ by (34), and $\langle pp_t\rangle = 0$ by (ii). Hence

$$W = 0 \quad \text{to order} \quad \left(\frac{q^2}{c}\frac{2}{\gamma\tilde{N}^{\frac{1}{2}}}\right). \tag{38}$$

As we shall find U of order q^2/c, this implies that

$$W \ll \gamma^{-\frac{1}{2}}\tilde{N}^{-1}U \quad \text{or} \quad W \ll 10^{-3}U.$$

From various biological and chemical considerations Knauss (1966) estimates W to be 0·1 to 1 m/day and upwards at the equator; the equatorial U is 1 m/sec eastward, so perhaps $W = +10^{-6}U$ to $+10^{-5}U$, certainly much less than $10^{-3}U$. It would be very interesting if a higher-order calculation would yield a definite result. One thing that has been neglected is the dependence of $N(z)$ on latitude resulting from the geostrophic adjustment of the density field to the mean current. We have also ignored the 'smaller' Coriolis terms in (1) and (3).

Horizontal flow

The expression (33) in terms of the non-dimensional variables now becomes

$$V = \beta^{-1}\mathbf{k}.(\nabla\times\boldsymbol{\tau}) = \frac{q^2}{c}\{\langle(uu_\xi+vv_\eta)_\eta-(uv_\xi+vv_\eta)_\xi\rangle Z_\zeta^2+\langle(up_t)_\eta-(vp_t)_\xi\rangle ZZ_{\zeta\zeta}\} \tag{39}$$

$$= -\frac{q^2}{c}\{\langle(u\omega)_\xi+(v\omega)_\eta\rangle Z_\zeta^2+\langle p_t\omega-up_{t\eta}+vp_{t\xi}\rangle ZZ_{\zeta\zeta}\}. \tag{40}$$

Taking advantage of the relation (35) pertaining to the vertical component of vorticity, (40) eventually reduces to

$$V = \frac{q^2}{c}\{\langle uv^t\rangle_\xi Z_\zeta^2-\langle vp\rangle(ZZ_\zeta)_\zeta\}, \tag{41}$$

where v^t denotes
$$\int^t v(\xi,\eta,t_1)\,dt_1.$$

For details see the appendix.

With $W=0$, (32) is simply $U_\xi+V_\eta=0$, which may be integrated to give

$$U = -\frac{q^2}{c}\{\langle uv^t\rangle_\eta Z_\zeta^2+\langle up\rangle(ZZ_\zeta)_\zeta\} \tag{42}$$

plus an arbitrary function of y, z.

We may now form vertical averages in accordance with (iv):

$$\overline{(ZZ_\zeta)_\zeta} \sim \overline{\sin\kappa z\cos\kappa z} = 0, \quad \overline{Z_\zeta^2} = \overline{(\tilde{N}(z)/\tilde{N})\cos^2\kappa z} \approx \tfrac{1}{2}\tilde{N}(z)/\tilde{N};$$

therefore

$$\left.\begin{aligned} \psi &= \frac{a}{\gamma^{\frac{1}{2}}}\frac{q^2}{c}\langle uv^t\rangle\overline{Z_\zeta^2}+F(\eta,\zeta), \\ \overline{U} &= -(\gamma^{\frac{1}{2}}/a)\psi_\eta, \quad \overline{V} = (\gamma^{\frac{1}{2}}/a)\psi_\xi, \end{aligned}\right\} \tag{43}$$

with $F(\eta,\zeta)$ an arbitrary function. Or, reverting to dimensional time and dimensional $u, v(x,y,z,t)$, we have simply

$$\psi = \langle uv^t\rangle + F(y,z). \tag{44}$$

In general

$$\psi = \sum_{n_1,n_2}\langle u_{n_1}v_{n_2}^t\rangle + F(y,z)$$

is the result of all possible interactions. Certain simple parity rules emerge:

	Integers		Functions			
	n_1	n_2	$u_{n_1}(\eta)$	$v_{n_2}(\eta)$	$\psi(\eta)$	$U(\eta)$
(i)	Odd	Odd	Even	Odd	Odd	Even (+)
(ii)	Even	Even	Odd	Even	Odd	Even (−)
(iii)	Odd	Even	Even	Even	Even	Odd
(iv)	Even	Odd	Odd	Odd	Even	Odd

But we have shown that only like integers interact upon reflexion. Thus, cases (iii) and (iv) are not 'reflexion coupled' (though they might be source coupled); accordingly, we expect the u-phase to be random with respect to the v-phase and the mean products $\langle uv^t \rangle$ to vanish. (To those who share our enthusiasm for the present hypothesis, the equatorial symmetry of the Cromwell current is evidence for poor odd–even coupling.)

For the simple cases we have been able to solve, U is positive (eastward) at the equator for even functions $u(\eta)$ and negative for odd $u(\eta)$, in the case of planetary waves; opposite in the case of gravity waves. The integral $\int v\,dt$ favours low frequencies in proportion to σ^{-1}. As a result, planetary waves are more effective current generators than gravity waves.

6. Some special cases

Simple progressive wave

Here we consider the case of an elementary train characterized by σ, n, s; s can be s^+ *or* s^-, but not both. From (14) and (17) we write simply (in dimensional units)

$$u(x,y,z,t) = q\cos(sx-\sigma t)\,Y^u Z_\zeta,$$
$$v(x,y,z,t) = -q\sin(sx-\sigma t)\,YZ_\zeta,$$
$$\psi = -\tfrac{1}{2}q^2\sigma^{-1}\,YY^u\overline{Z_\zeta^2}+F(y,z).$$

Thus $V = \psi_x = 0$, $U_x = -V_y = 0$, and U is an undetermined function of Y and Z and has nothing to do with the planetary waves. The Kelvin wave solution treated in § 3 falls into this category.

Oppositely travelling free waves, no barriers

We visualize a source generating s^+ *and* s^- (associated with negative and positive group velocity) that travels many times around the earth. For any single frequency σ and mode n, we have

$$\left.\begin{aligned} u &= [q^+\cos(s^+x-\sigma t+\theta^+)\,Y^{u+}+q^-\cos(s^-x-\sigma t+\theta^-)\,Y^{u-}]Z_\zeta, \\ v &= -[q^+\sin(s^+x-\sigma t+\theta^+)+q^-\sin(s^-x-\sigma t+\theta^-)]\,YZ_\zeta, \end{aligned}\right\} \tag{45}$$

with the phases θ^+ bearing a fixed relation to θ^- owing to the common origin of the $s^\pm$ wave-numbers. The result is

$$\psi = -\frac{1}{2\sigma}[q^+q^-\cos(2s'x+\theta^+-\theta^-)(Y^{u+}+Y^{u-})+(q^+)^2Y^{u+}+(q^-)^2Y^{u-}]\,Y\overline{Z_\zeta^2} +F(y,z), \tag{46}$$

thus leading to *cellular* streaming with $2s'$ cycles around the earth.

An interior barrier

Now visualize a knife-edge barrier at x_1, y_1 so that $u = 0$ and $U = 0$ at x_1, y_1. From (45) we find that this condition requires

$$\theta^+ = \theta^- \quad \text{and} \quad q^+ Y^{u+} = -q^- Y^{u-} = q, \quad \text{say,}$$

and from (46)

$$\psi = -\frac{q^2}{\sigma} \sin^2 s'(x-x_1) \frac{Y^{u+}(y) + Y^{u-}(y)}{Y^{u+}(y_1) \, . \, Y^{u-}(y_1)} Y(y) \overline{Z_\xi^2} + G(y,z),$$

with $G_y(y_1, z) = 0$. The barrier produces *rectified* (sign independent of x) streaming at latitude y_1, with the direction of the streaming depending on the Y-functions. The algebraic basis of this peculiar result is that

$$u \sim \sin s'(x-x_1), \langle uv^i \rangle \sim \sin^2 s'(x-x_1)$$

at y_1; hence, with $G_y(y_1) = 0$ we find that

$$\psi_y, U \sim \sin^2 s'(x-x_1)$$

are rectified. The result suggests a radical modification of streaming when a knife is stuck into a suitable rotating fluid. With regard to the oceans, we might look for a steady zonal flow to the east and west of islands. When there are many barriers, we run into the host of difficulties that have been discussed in connexion with standing waves between boundaries.

Standing waves between boundaries

The simplest possible case is that of

$$n' = n'' = n, \tag{47}$$

so that there is only one term in the n-summation yielding a real s. The prerequisite conditions follow from (28) and (24):

$$\sin s'_n \xi_L = 0 \quad \text{and} \quad 2n+1 < \sigma^2 + \tfrac{1}{4}\sigma^{-2} < 2n+5, \tag{48a, b}$$

where s' and σ are now Blandford–Matsuno frequencies. The non-dimensional solution is

$$u(\xi,\eta,\tau) = \Bigg\{\exp\{i(s_n^+\xi - \sigma\tau)\}\left[\frac{(\frac{1}{2}n)^{\frac{1}{2}}}{s_n^+ + \sigma} Y_{n-1} - \frac{\{\frac{1}{2}(n+1)\}^{\frac{1}{2}}}{s_n^+ - \sigma} Y_{n+1}\right]$$

$$- \frac{s_n^- + \sigma}{s_n^+ + \sigma} \exp\{i(s_n^-\xi - \sigma\tau)\}\left[\frac{(\frac{1}{2}n)^{\frac{1}{2}}}{s_n^- + \sigma} Y_{n-1} - \frac{\{\frac{1}{2}(n+1)\}^{\frac{1}{2}}}{s_n^- - \sigma} Y_{n+1}\right]\Bigg\}$$

$$= \frac{(2n)^{\frac{1}{2}}}{s_n^+ + \sigma} \sin(s^0\xi - \sigma\tau) \sin s'_n \xi Y_{n-1} + \ldots + Y_{n+1} + \text{boundary layer}, \tag{49}$$

$$v(\xi,\eta,\tau) = \frac{-2}{s_n^+ + \sigma}[(s^0+\sigma)\cos(s^0\xi - \sigma\tau)\sin s'_n \xi + s'_n \sin(s^0\xi - \sigma\tau)\cos s'_n \xi]\, Y_n$$

$$+ \text{boundary layer,}$$

$$\langle uv^\tau \rangle = -\frac{(2n)^{\frac{1}{2}}}{\sigma} K \sin^2 s'_n \xi \quad Y_{n-1} Y_n + \ldots Y_{n+1} Y_n + \text{boundary layer,}$$

where

$$K = \frac{(-s^0) - \sigma}{(s_n^+ + \sigma)^2}.$$

The u-component in Y_{n+1} does not vanish at $\xi = 0, \xi_L$ without including the first boundary-layer terms, and we shall ignore Y_{n+1} for the time being. The result is

$$\psi = -\frac{(q^+)^2}{\sigma_d}(2n)^{\frac{1}{2}}K\sin^2 s'_n x \quad \overline{Y_{n-1}Y_nZ^2_\xi} \tag{50}$$

in dimensional form, except that K is in terms of Blandford–Matsuno frequencies. Had we included the complete set of Y_{n+1} terms, there would have been a correction factor $1+2\sigma^2$, and an additional term which in the interior has a simple $\cos^2 s'_n x$ dependence on x. Using the asymptotic forms of (25) and the restriction (48*b*), it can be shown that the factor $(q^+)^2K/\sigma_d$ favours planetary waves over gravity waves as a source of streaming in the ratios (for large n) $4n$ and $64n$, respectively, depending on whether the two wave classes have the same u-components or the same v-components. The direction of streaming is opposite for the two classes of waves, with the sign reversal occurring at the critical point

$$\sigma = 2^{-\frac{1}{2}}, s^0 = -1/2\sigma = -2^{-\frac{1}{2}} \text{ (B.M. frequencies).}$$

At the eastern side of the Pacific, the equator is blocked by the Galapagos Islands, and Knauss (1966) finds a reduced current at the station located at 96° W., 400 km seaward of the Galapagos. The theory indicates a boundary layer of width $\sim a\gamma^{-\frac{1}{2}} = 350r^{-\frac{1}{2}}$ km, equation (29). At least there is no conflict here. On the western side neither the measurements nor the boundary are definitive.

The y-dependence of U is given by the term $(Y_{n-1}Y_n)_\eta$, which, aside of normalizing factors, can be written

$$(Y_{n-1}Y_n)_\eta \sim \exp(-\eta^2)[-2\eta H_nH_{n-1} + (H_nH_{n-1})\,\eta].$$

The most important problem here is the sign of U at the equator. For Rossby solutions, $\sigma < 2^{-\frac{1}{2}}$, U is positive (eastward, as observed), when (referring to equation (49)) $u(\eta)$ is an even function and $v(\eta)$ an odd function. In order for our hypothesis to be valid, an argument will have to be made for a source mechanism that favours equatorially symmetric Rossby waves, that is, $u(\eta) = u(-\eta)$.

One might attempt to gain some insight into the expected profile away from the equator by superimposing solutions (50):

$$\psi \sim \sum_{n=1}^{\infty} F(n)Y_{n-1}Y_n.$$

The procedure ignores all n-interactions except those with its neighbours; it remains to be seen whether the argument for rectification can be maintained in the more general case. A serious limitation is imposed by the r-dependent scaling, for this obscures the role of a variable r-content. We proceed, nevertheless, using the 'equatorial approximation'

$$Y_n \sim n^{-\frac{1}{4}}\cos(2n+1)^{\frac{1}{2}}\eta + O(n^{-\frac{3}{4}}) \quad (\eta^2 \ll 2n+1),$$

and this gives

$$\psi \sim -\sum_{n=1}^{n_L} Y_{n-1}Y_n dn = -\int_0^{n_L} n^{-\frac{1}{2}}\sin(2(2n)^{\frac{1}{2}}\eta)\,dn$$
$$= -\eta^{-1}\sin^2((2n_L)^{\frac{1}{2}}\eta). \tag{51}$$

At the equator, $\psi = -2n_L\eta$ and $U \sim -\psi_\eta = 2n_L$. The width of the equatorial 'jet' is given roughly by $(2n_L)^{\frac{1}{2}}\eta_c = \frac{1}{2}\pi$, or

$$\phi_c^2 = \frac{\pi^2}{8}\frac{1}{n_L\gamma}. \tag{52}$$

Beyond η_c, equation (51) shows a typical $\sin^2 x/x$ diffraction pattern owing to the assumed abrupt cut-off at n_L. For a smooth fading function the behaviour of $\psi(\eta)$ beyond η_c would be more like half the envelope, $\psi(\eta) \sim -\eta^{-1}$ and $U = -\psi_\eta = -\eta^{-2}$. The ratio of the return flow to the equatorial jet is of the order

$$\frac{\eta_c}{\eta_T} = \frac{(2/n_L)^{\frac{1}{2}}}{(2n_L+1)^{\frac{1}{2}}} \approx \frac{1}{n_L}.$$

The present treatment suggests a return flow in subtropical latitudes.

Finally, the z-dependence $\overline{Z_\zeta^2} \sim N(z)$ is the same as previously discussed.

7. Discussion

We have interpreted the submerged, equatorial profile of the Cromwell current in terms of the baroclinic, equatorial steering of internal planetary waves. The second-order theory gives a rectified component $U \sim q^2/c$ and to fit observed data the three velocities here involved, U, q, c, might all be of the order of 1 knot. In the few cases where adequate time series of ocean currents were observed, it was found that the fluctuations were of the same order as the mean current. Perhaps the Cromwell current is no exception; clearly what is needed is measurements of planetary waves at the equator. At the moment we have made the dubious contribution of explaining a well-observed effect by an unobserved cause.

Our interpretation $q \sim c$ conflicts with the assumption implicit in the perturbation treatment, and we conveniently ignore all problems involving sum frequencies, multiple interactions, forcing functions, etc.

Finally, it should be clearly stated that our results are anticipated by any of the previous investigations that emphasize the role of Reynolds stresses (Robinson 1965). Perhaps our contribution is to state the problem in more specific terms. At first hand it seems unreasonable to ascribe to a long-period and relatively weak (if it exists at all) wave motion an intense rectification and associated non-linear effects usually reserved for 'finite amplitude' waves. But, in fact, the essential parameter is amplitude/length, and this can be large also if the wavelength is small. For Rossby waves the 'inverse dispersion' ($\partial\sigma/\partial s$ negative) leads to small wavelengths at large T, and so Rossby waves are well rectified under the unaccustomed conditions of small amplitude and large period.

We are reminded of Pedlosky's (1965) remarkable conclusion that the rectified transient currents associated with the ocean's response to variable winds are a *significant* component of the Gulf Stream system. If this is the case in a region where the mean wind-driven circulation is greatly intensified and the Rossby waves not particularly channelled, would rectification not be *predominant* at the equator where the wind effects are not particularly emphasized, and Rossby waves are uniquely concentrated?

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Appendix

We use the following steps to simplify equation (40):

1. $\langle up_{t\eta} - vp_{t\xi} \rangle = -\langle u_t p_\eta - v_t p_\xi \rangle$ by (34). Substituting for p_ξ and p_η from (14) we find

$$-\langle up_{t\eta} - vp_{t\xi} \rangle = \langle u_t(v_t - \eta u) - v_t(u_t - \eta v) \rangle = \tfrac{1}{2}\langle \eta (u^2 + v^2)_t \rangle = 0 \qquad \text{(A 1)}$$

by §5 (ii).

2. Multiply (36) by p and time-average to obtain $\langle \omega_t p \rangle + \langle vp \rangle = 0$, or

$$\langle \omega p_t \rangle = \langle vp \rangle. \qquad \text{(A 2)}$$

3. Integrate (36) with respect to time to obtain

$$\omega = -v^t + \eta p. \qquad \text{(A 3)}$$

4. Substitute for w from (A 3) to obtain

$$\begin{aligned}(u\omega)_\xi + (v\omega)_\eta &= (-uv^t + \eta up)_\xi + (-vv^t + \eta vp)_\eta \\ &= -\langle uv^t \rangle_\xi + \eta \langle up \rangle_\xi - \langle vv^t \rangle_\eta + \langle vp \rangle + \eta \langle vp \rangle_\eta \\ &= -\langle uv^t \rangle_\xi + \langle vp \rangle + \eta(\langle up \rangle_\xi + \langle vp \rangle_\eta) - \langle vv^t \rangle_\eta \\ &= -\langle uv^t \rangle_\xi + \langle vp \rangle, \end{aligned} \qquad \text{(A 4)}$$

since $\langle up \rangle_\xi + \langle vp \rangle_\eta = 0$ by (35), and $\langle vv^t \rangle = 0$ by §5 (ii).

5. Substituting from (A 1), (A 2) and (A 4) into (40) gives (41).

REFERENCES

BLANDFORD, R. 1966 Mixed gravity–Rossby waves in the ocean. *Deep-Sea Res.* **13**, 941–61.

COX, C. S. & SANDSTROM, H. 1962 Coupling of internal and surface waves in water of variable depth. *J. Oceangr. Soc. Japan*, 20th Anniv. Vol.

ERDELYI, 1953 *Higher Transcendental Function*, vol. II. New York: McGraw-Hill.

GROVES, G. 1965 Doctoral thesis. Scripps Institution of Oceanography.

KNAUSS, J. A. 1960 Measurements of the Cromwell current. *Deep-Sea Res.* **6** (4), 265–85.

KNAUSS, J. A. 1966 Further measurements and observations on the Cromwell current. *J. Mar. Res.* **24**, 205–40.

LONGUET-HIGGINS, M. S. 1964 Planetary waves on a rotating sphere, I. *Proc. Roy. Soc.* A **279**, 446–73.

LONGUET-HIGGINS, M. S. 1965 Planetary waves on a rotating sphere, II. *Proc. Roy. Soc.* A **284**, 40–54.

MATSUNO, T. 1966 Quasi-geostrophic motions in the equatorial area. *J. Met. Soc. Japan* II, **44**, 25–43.

METCALF, W. G., VOORHIS, A. D. & STALCUP, M. C. 1962 The Atlantic equatorial undercurrent. *J. Geophys. Res.* **67**, 2499–508.

MONTGOMERY, R. B. & STROUP, E. D. 1962 Equatorial water and currents at 150° west in July–August 1952. *Johns Hopkins Oceanogr. Stud.* I, 68 pp.

PEDLOSKY, J. 1965 A study of the time dependent ocean circulation. *J. Atmos. Sciences* **22**, 267–72.

PHILLIPS, N. 1966 Large-scale eddy motion in the Western Atlantic. *J. Geophys. Res.* **71**, 3883–91.

RATTRAY, M. & CHARNELL, R. L. 1966 Quasi-geostrophic free oscillations in enclosed basins. *J. Mar. Res.* **24**, 82–103.

REID, J. L. 1964 A transequatorial Atlantic oceanographic section in July, 1963 compared with other Atlantic and Pacific sections. *J. Geophys. Res.* **69**, 5205–16.

ROBINSON, A. R. 1965 Chapter 17 of *Research Frontiers in Fluid Dynamics*, pp. 504–33. New York: Interscience.

ROBINSON, A. R. 1966 An investigation into the wind as the cause of the equatorial undercurrent. *J. Mar. Res.* **24**, 179–204.

SVERDRUP, H. U. 1947 Wind-driven currents in a baroclinic ocean, with application to the equatorial currents of the eastern Pacific. *Proc. Natn. Acad. Sci. U.S.A.* **33**, 318–26.